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Bergmannian Relativity and Bracket Spaces

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We explore an avenue of higher-dimensional spacetimes based on generalized spinors which transform under the special linear groups and result in spacetime dimensions which are squares of integers. The Bergmannian chronometrics are not Riemannian, but Finslerian in the higher dimensions. The general concept of bracket space is introduced in order to show a variety of routes to hyperspace. The field equations found generalize Einstein's by replacing a factor of two by the spinorial dimension. A mass term is introduced in the action, which results in a hyper-stress-energy-momentum tensor. The chronometric is not required to be covariantly constant under the hyper-Palatini variations: there is torsion. "Spherical" symmetry in this spacetime is explored, an appropriate set of coordinates is introduced, and the invariant for nine-dimensional "spherical" symmetry is given.

1. INTRODUCTION

The Quantum Topology Workshop at Georgia Tech has pursued the possibility of describing higher-dimensional spacetimes using quantum spinors (Finkelstein *et al.*, 1985, 1987; Finkelstein, 1987; Holm, 1986, 1987, 1989). Section 2 of this paper treats the Bergmannian chronometric as an alternative to the Riemannian chronometric. The concept of a bracket space is introduced in Section 3 to connect several concepts of hyperspace in a logical fashion in Section 4.

The four subsections of Section 5 are devoted to one sequence of hyperspaces suggested by the Bergmannian chronometric. Section 5.1 describes how the chronometric is placed on the spacetime. Section 5.2 lists assumptions used in general relativity and then describes how the list has changed due to this particular higher-dimensional generalization of general relativity. Section 5.3 evaluates a few possibilities for raising and lowering

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tensor indices. Section 5.4 contains the derivation of the higher-dimensional equation of motion for this spacetime, obtained by varying an action. The action chosen is similar to that of general relativity. The hyper-stress-energy-momentum tensor is introduced in Section 5.4 and the Newtonian limit is examined to find the kappa coefficient. The hyper-Palatini method is introduced and yields a different result than the usual chronometric constancy.

In Section 6 we generalize the Schwarzschild spherically symmetric spacetime. The coordinates include one time axis, $N-1$ radial scalars, and $N(N-1)$ angles. The last subsection of Section 5 examines the $N=3$ case, i.e., nine-dimensional "spherically" symmetric spacetime, and an invariant spacetime interval is found.

2. CHRONOMETRICS IN HYPERSPACE

2.1. Einstein Chronometric

In special relativity $d\tau^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2$ is the invariant interval between spacetime events. From here on we set $c=1$ to simplify the equations. The invariant for special relativity is often written as

$$d\tau^2 = \eta_{ab}\, dx^a\, dx^b$$

where $a, b = 0, 1, 2, 3$ and where $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$, all other components are zero, and $dx^0 = dt$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$. This matrix, η, is called the Minkowski chronometric.

A chronometric describes the spacetime interval, while a metric in the topological sense describes the distance between two points of space. A chronometric is not a metric. However, when the meaning is clear in context, "chronometric" is abbreviated by "metric."

In curved spacetime the interval is generalized to allow cross terms and to allow functions of spacetime in front of the terms. Then the invariant is of the form $d\tau^2 = g_{ab}\, dx^a\, dx^b$, where $a, b = 0, 1, 2, 3$. The g_{ab} are then components of the Einstein chronometric. These components are the real coefficients which describe the gravitational field. Due to the form of $d\tau^2$, we can let $g_{ab} = g_{ba}$ without loss of generality.

2.2. Riemannian Chronometric

The n-dimensional Riemannian extension of curved spacetime is defined by setting a and b to range from zero to $n-1$ instead of 3, while keeping the form of the invariant $d\tau^2 = g_{ab}\, dx^a\, dx^b$, where $a, b = 0, 1, \ldots, n-1$. Then the g_{ab} are components of the Riemannian chronometric. The Riemannian extension of curved spacetime seems natural when the invariants are presented as in the previous section. An example of the Riemannian extension is the Kaluza–Klein extension to $n=5$ to include electromagnetism, where $g_{50}=A_0$ and $g_{5i}=A_i$ and where A is the electromagnetic potential (A is a gauge field). Often the higher dimensions are used to include gauge fields in curved spacetimes.

2.3. Spin Form

The chronometric is taken as the fundamental variable of spacetime for relativities based on the Riemannian chronometric. We shall take as the fundamental variable a tensor object called the spin form. The following discussion on vectors, tensors, spinors, and spin forms is in preparation for the section on the Bergmannian chronometric.

For every vector space V, there exists a dual vector space V^D, a complex conjugate vector space V^*, and a complex conjugate dual vector space V^{*D}. Denote the components of elements of V by x^a; those of V^D by x_a; those of V^* by $x^{a'}$; and those of V^{*D} by $x_{a'}$. These spaces can be defined as follows:

1. $V^D=\{f \mid f\colon V\to\mathbb{C}; f \text{ linear}\}$
2. $V^{*D}=\{f \mid f\colon V\to\mathbb{C}; f \text{ antilinear}\}$
3. $V^*=\{f \mid f\colon V^{*D}\to\mathbb{C}; f \text{ linear}\}$

If you think of V as column vectors, then think of V^D as row vectors, V^* as column vectors, and V^{D*} as row vectors. Recall that a row vector times a column vector equals a number.

A tensor in

$$V^D\otimes\cdots\otimes V^D\otimes V\otimes\cdots\otimes V\otimes V^{*D}\otimes\cdots\otimes V^{*D}\otimes V^*\otimes\cdots\otimes V^*$$

can be defined as an element of the class

$$T=\{f \mid f\colon W\to\mathbb{C}; f \text{ multilinear}\}$$

where

$$W=V\times\cdots\times V\times V^D\times\cdots\times V^D\times V^*\times\cdots\times V^{*D}\times\cdots V^{*D}$$

T is said to be a tensor of type $(p, q; r, s)$, where p is the number of V^D's, q the number of V's, r the number of V^{*D}'s and s the number of V^*'s in W. Thus, a tensor component has indices which are raised, lowered, raised primed, and lowered primed. The components of a tensor are dependent on the basis vectors, basis dual vectors, ... Changing from one basis set to another yields a transformation of the tensor components.

Let flat spacetime be represented relative to an origin by a real n-dimensional vector space V_n with vector elements represented by components

x^a (i.e., small indices). Similarly, let the spin space be represented by a complex N-dimensional vector space S_N with spin-structure elements represented by components x^A (i.e., capital indices). The homomorphism between the Lorentz group and the spin group is such that the homogeneous Lorentz group is doubly covered by the spin group, which implies that the product $S_N \times S_{N*}$ corresponds to an N^2-dimensional vector space. The components of tensors on $S_N \times S_{N*}$ are represented by $x^{AA'}$. In Section 4, we shall require $x^{AA'}$ to be Hermitian. Then $x^{AA'}$ will have fewer independent components than two full spinors. For this reason one calls $x^{AA'}$ a sesquispinor (one and a half spinors). The components $x^{AA'}$ of the sesquispinor can be mapped to a component x^a of the vector by

$$x^{AA'} = \sigma_a{}^{AA'} x^a$$

where $\sigma_a{}^{AA'}$ is a spin form.

2.4. Bergmannian Chronometric

Since spin is basic to the structure of quantum mechanics, we look to spin as the basic structure of relativity in the desire to combine both concepts (quantum mechanics/relativity) into one theory.

One way to represent the spin form in four dimensions is by using the 2×2 identity matrix $\mathring{\sigma}_0$ and the three Pauli matrices $\mathring{\sigma}_1$, $\mathring{\sigma}_2$, $\mathring{\sigma}_3$. A sesquispinor X may be expressed as $X = t\mathring{\sigma}_0 + X_S$, where $X_S = x\mathring{\sigma}_1 + y\mathring{\sigma}_2 + z\mathring{\sigma}_3$. Then the invariants for Newtonian physics are given by $dx^2 + dy^2 + dz^2 = \det(dX_S)$ and $dt = \frac{1}{2}$ trace(dX). The invariant for special relativity is

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = \det(dX)$$

In curved spacetime this concept can be extended to any linearly independent set of four 2×2 Hermitian matrices represented by the symbols σ_0, σ_1, σ_2, σ_3, with $X = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3$. Then the curved spacetime invariant is (Penrose and Rindler, 1986)

$$d\tau^2 = g_{ab}\, dx^a\, dx^b = \det(dX)$$

The N^2-dimensional extension of curved spacetime which we consider here uses a linearly independent set of $N \times N$ Hermitian matrices with indices ranging from zero to $N^2 - 1$. The invariant for this new space is

$$d\tau^N = \det(dX) = g_{abc\ldots}\, dx^a\, dx^b\, dx^c \ldots$$

where g has N indices. The $g_{abc\ldots}$ are called components of a Bergmannian chronometric. This chronometric is symmetric and linear. The $g^{abc\ldots}$ are called components of the inverse Bergmannian chronometric. Although the Riemannian and Bergmannian chronometrics are equivalent for four-dimensional spacetime, they are distinct for higher dimensions.

3. BRACKET SPACES

3.1. Introduction

General relativity is often expressed in tensor formalism, or in spinor formalism, which includes the tensor formalism and is more fundamental. There are three higher-dimensional spin structures which reduce to general relativity for $n=4$: Riemannian spinors, Bergmannian spinors, and symplectic spinors. Each of these three spin structures can be used to extend general relativity from four dimensions to higher dimensions. To see the difference between the spin structure approaches, we introduce the unifying concept of the bracket space which is used to define the associated spin structure. Each spin structure follows from a particular bracket space. The spin structure depends on a bracket group and a bracket algebra. It is important that one stays within one bracket space when building a spin structure.

3.2. Bracket Space

A bracket space $\{V, [\ ], A\}$ is defined by the following three structures.

1. A vector space V (typically $\mathbb{R}^n$ or $\mathbb{C}^n$) over a field F (typically $\mathbb{R}$ or $\mathbb{C}$).
2. A bracket [ ]. Let $kV \equiv V \oplus \cdots \oplus V$, where there are k vector spaces entering the direct sum. A general element of kV shall be designated by B. A bracket [ ] is a map $f: kV \to F$ linear in each factor V (i.e., multilinear in V). We call k the degree of the bracket.

The group of a bracket [ ] is the group $G_{[\ ]}$ of linear transformations of V that preserve the bracket, i.e., for any linear transformation M, $\forall x_i \in V$, $f(Mx_1, \ldots, Mx_k) = f(x_1, \ldots, x_k)$. Once the bracket is chosen, the group is determined.

Let S_k be the symmetric group acting on the k vectors of kV. We postulate the existence for each $x \in S_k$ of a semilinear (i.e., linear or antilinear) map

$$\beta(x):\quad kV \to kV$$

with

$$\beta(xy) = \beta(x)\beta(y)$$

$$\beta(1) = 1$$

and satisfying the bracket relation

$$[x\beta(x)B] = [B]$$

where x is an element of S_k.

3. A linear free associative algebra A. This is defined when a bilinear product $\odot$ is given to the vector space V, such that

$$\forall x_1, x_2 \in V | x_1 \odot x_2 \in A$$

and

$$\forall x, y, z \in A', \quad x \odot (y \odot z) = (x \odot y) \odot z$$

where $\odot$ is distributive over addition.

Let Π be the multilinear product operator

$$\Pi: \quad kV \to A'$$

$$\Pi B = \Pi(x_1 \oplus \cdots \oplus x_k) = x_1 \odot \cdots \odot x_k$$

Define the symmetrizing operator $\Sigma_{[\,]}$ which operates on an ordered k-tuple to result in a sum of ordered k-tuples which have the same symmetry as the bracket, by

$$\Sigma_{[\,]} B \equiv \frac{1}{k!} \sum_{x \in S_k} x\beta(x) B$$

It can be shown that

$$(\Sigma_{[\,]})^2 = \Sigma_{[\,]}$$

We postulate that for some $\varepsilon \in A$ and all $\beta \in kV$ and all $a \in A$,

$$\Pi\Sigma_{[\,]} B = [B]\varepsilon$$

$$\varepsilon a = a\varepsilon$$

Then $\{V, [\ ], A\}$ is a bracket space.

The bracket space thus has associated with it the following auxiliary structures as well:

S_m	a symmetric group
$\beta(x)$	a bracket conjugation
$G_{[\,]}$	a bracket group
ε	an element of the center of A

The following are four examples of bracket spaces, three of which we will use to build associated spin structures.

3.2.1. Orthogonal Bracket Space

Let V be a real Hilbert space, i.e., $V = \mathbb{R}^n$, $F = \mathbb{R}$, $[\ ] =$ the dot product. An $n \times n$ square matrix has signature $\langle \alpha, \beta \rangle$ if it can be transformed by a similarity transformation to a diagonal matrix which has α entries $+1$, β entries -1, and $(n-\alpha-\beta)$ entries zero. Every real symmetric matrix can be transformed in this way. Let S be a nondegenerate symmetric matrix with signature $\langle k, n-k \rangle$. The dot product is given by $f(x, y) = \{x, y\} \equiv x^T S y$. Thus,

$$f(Mx, My) = (Mx)^T S(My) = x^T(M^T SM)y$$

So the group associated with the dot product is the group of all matrices for which $M^T SM = S$, the orthogonal group $O(k, n-k, \mathbb{R})$. Notice that $O(a, b, \mathbb{R})$ is the same group as $O(b, a, \mathbb{R})$. The dot product is symmetric; thus the permutation subgroup is $S_2 = (1, X)$, represented by $\beta(x) = \text{identity}$. Thus we generate an algebra by

$$\Pi\Sigma_+(x_1 \oplus x_2) = \{x_1, x_2\}e$$

which can also be written as

$$\tfrac{1}{2}(x \odot y + y \odot x) = \{x, y\}e$$

where e is defined as the identity. This is the (orthogonal) Clifford algebra.

Thus the orthogonal bracket space is given by

$$\{\mathbb{R}^n, \text{dot product, Clifford algebra}\}$$

with auxiliary structures

$$\{S_2, \text{identity, orthogonal group, identity}\}$$

3.2.2. Symplectic Bracket Space

Let V be a symplectic space, i.e., $V = \mathbb{R}^{2n}$, $F = \mathbb{R}$, [ ] = symplectic bracket. Let J be an $n \times n$ nonsingular antisymmetric matrix. The symplectic bracket is given by $[x, y] \equiv x^T J y$. Thus,

$$f(Mx, My) = (Mx)^T J(My) = x^T(M^T JM)y$$

So the group associated with the symplectic form is the group of all matrices for which $M^T JM = J$, the symplectic group $Sp(n, \mathbb{R})$. [Do not confuse the symplectic group with the infinitesimal symplectic group, the group of all matrices K satisfying the relation $K^T J + JK = 0$ or equivalently $(e^{sK})^T J(e^{sK}) = J$ for all numbers s.] The symplectic bracket is antisymmetric; thus the auxiliary symmetric group is $S_2 = (1, X)$, represented by $\beta(1) = 1$, $\beta(X) = -1$, which we write as $\beta(x) = (-1)^x$. Thus, we generate an algebra by

$$\Pi\Sigma_-(x_1 \oplus x_2) = [x_1, x_2]e$$

which can also be written as

$$\tfrac{1}{2}x \odot y - y \odot x = [x, y]e$$

where e is the identity. This is the Weyl algebra (or the symplectic Clifford algebra).

Thus, the symplectic bracket space is

$$\{\mathbb{R}^n, \text{symplectic bracket, Weyl algebra}\}$$

with auxiliary structures

$$\{S_2, (-1)^x, \text{symplectic group, identity}\}$$

3.2.3. Hilbert Bracket Space

Let V be a complex Hilbert space, i.e., $V=\mathbb{C}^N$, $F=\mathbb{C}$, $[\ ]=\langle\ \rangle=$ inner product. The inner product is given by $[x, y]=\langle x, y\rangle \equiv x^\dagger y$. Thus,

$$[Mx, My]=(Mx)^\dagger(My)=x^\dagger(M^\dagger M)y$$

So the group associated with the inner product is the group of all matrices for which $M^\dagger M=1$, the unitary group $U(n, \mathbb{C})$. The inner product is complex conjugate symmetric; thus, the permutation subgroup is $S_2=\{1, X\}$, with $\beta(1)=1$, $\beta(X)=$ Hermitian conjugate $=\dagger$, written $\beta(x)=\dagger^x$. Thus, we generate an algebra by

$$\Pi\Sigma_H(x_1{}^H \oplus x_2)=\langle x_1, x_2\rangle e$$

which can also be written as

$$\tfrac{1}{2}(x \odot y^* + y^* \odot x)=\langle x, y\rangle e$$

where e is defined as the identity. We call this algebra the Dirac–Hilbert algebra.

Thus, the Hilbert bracket space is given by

$$\{\mathbb{C}^n, \text{inner product, Dirac–Hilbert algebra}\}$$

$$\{S_H, \dagger^x, \text{unitary group, identity}\}$$

3.2.4. Grassmann Bracket Space

Let V be a Grassmann space, i.e., $V=\mathbb{C}^N$, $F=\mathbb{C}$, $[\ ]=$ Grassmann bracket. Let $x_1, \ldots, x_N$ denote the determinant composed of the vectors $x_1, \ldots, x_N$ as columns. The Grassmann bracket is given by

$$[x_1, \ldots, x_N]=|x_1, \ldots, x_N|$$

Thus,

$$\begin{aligned}[Mx_1, \ldots, Mx_N] &= |Mx_1, \ldots, Mx_N| \\ &= (\det M)^N |x_1, \ldots, x_N| \\ &= (\det M)^N [x_1, \ldots, x_N]\end{aligned}$$

So the group associated with the Grassmann bracket is the group of all matrices M for which $(\det M)^N = 1$. The subgroup which is connected with the identity is called the special linear group, $SL(N, \mathbb{C})$. The Grassmann bracket has the symmetry $(-1)^p := 1$ for even permutations p, -1 for odd permutations p. Thus, the symmetric group is S_N, represented by $\beta(x) = (-1)^x$. Thus, we generate an algebra by

$$\Pi\Sigma_-(x_1 \oplus \cdots \oplus x_N) = [x_1, \ldots, x_N]E$$

which can also be written as

$$x_i \vee x_j + x_j \vee x_i = 0$$

and

$$x_1 \vee \cdots \vee x_N = [x_1, \ldots, x_N]E$$

where E is a top (defined below). This algebraic product is called the join. If the identity were used in place of the top, then the bracket algebra in this case would be the field algebra.

There exists a basis $\{b_1, \ldots, b_N\}$ such that $[b_1, \ldots, b_N] = 1$. Choose such a basis and define $E = b_1 \vee \cdots \vee b_N$, called top.

The join $x \vee y$ has a natural companion called the meet $x \wedge y$ defined by

$$\begin{aligned}(x_1 \vee \cdots \vee x_k) \wedge (x_{k+1}k \vee \cdots \vee x_m) &= [x_1, \ldots, x_m] \qquad \text{for } m = N \\ &= 0 \qquad \text{for } m < N\end{aligned}$$

The top E is an identity for the meet. The algebra with this combination of join and meet is called the Grassmann double algebra (Barnabei *et al.*, 1985).

Thus the Grassmann bracket space is given by

$$\{\mathbb{C}^N, \text{Grassmann bracket, Grassmann algebra}\}$$

with auxiliary structures

$$\{S_N, (-1)^x, SL_N, \text{top}\}$$

4. PATHS TO HYPERSPIN

4.1. Spin Structures

A vector space and a spinor space are associated through the spin form. We discuss elements of three spin structures which are used in physics to investigate higher-dimensional relativity. These three cases are called Riemannian, Bergmannian, and symplectic. For the Riemannian and symplectic cases the spinor space is constructed from the vector space, whereas in the Bergmannian case the vector space is constructed from the spinor space. Each structure uses constructs from one of the bracket spaces.

4.2. Riemannian

The Riemannian approach to spacetime goes via the orthogonal bracket space. An n-dimensional real vector space is used to generate a Clifford algebra of order n, C_n, which has

$$\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} = \sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ elements}$$

(Budinich and Trautman, 1988). The elements of this algebra are then used to construct a spin space. The dimension of this spin space is on the order of the square root of the algebra dimension,

$$N = 2^{n/2} \qquad \text{for } n \text{ even}$$
$$N = 2^{(n-1)/2} \qquad \text{for } n \text{ odd}$$

The symmetric dot product suggests we define

$$\varepsilon_{AB}\varepsilon_{A'B'}\sigma_a^{AA'}\sigma_b^{BB'} = g_{ab}$$

where ε_{AB} is an antisymmetric tensor, A and B range from zero to $N-1$, and a and b range from zero to $n-1$. Then the tensor g_{ab} is called the Riemannian chronometric. This spin structure is commonly used in higher-dimensional relativity. Since this case is the most common, the term "spinor" is often used, implying the term "Riemannian spinor." The context will tell which spin structure is being used.

4.3. Bergmannian

The Bergmannian approach to spacetime is via the Grassmann bracket space. An N-dimensional spinor space is used to directly construct a vector space which has the squared dimension,

$$n = N^2$$

The antisymmetric Grassmann bracket suggests we define

$$\frac{1}{(n-1)!}\,\varepsilon_{A\cdots Z}\varepsilon_{A'\cdots Z'}\sigma_a^{AA'}\cdots\sigma_z^{ZZ'} = g_{a\cdots z}$$

where the N-dimensional Levi-Civita symbol $\varepsilon_{A\cdots Z}$ is zero if there are repeated indices, one for $\varepsilon_{12\cdots N}$, and changes sign under index permutation. $A, \ldots, Z$ and $A', \ldots, Z'$ range from zero to $N-1$, and $a, \ldots, z$ range from zero to $n-1$. Then the tensor $g_{a\cdots z}$ is called the Bergmannian chronometric. This spin structure is used in Sections 5 and 6.

4.4. Symplectic

The symplectic approach to spacetime is via the symplectic bracket space. An n-dimensional vector space is used to generate a Weyl algebra of order n, C_n (n even), which has

$$\sum_{k=0}^{\infty}\frac{(n+k-1)!}{k!\,(n-1)!} = \infty \text{ elements}$$

although often the sum is truncated at some maximum value of k in order to keep the algebra finite. The sum suggests that the spinor space could be infinite, unless an appropriate cutoff is found. Symplectic spinors are also called spinsters. The symplectic product suggests we define both

$$\varepsilon_{AB}\varepsilon_{A'B'}\sigma_a^{AA'}\sigma_b^{BB'} = g_{ab}$$

and

$$\frac{1}{(N-1)!}\,\varepsilon_{A\cdots Z}\varepsilon_{A'\cdots Z'}\sigma_a^{AA'}\cdots\sigma_z^{ZZ'} = g_{a\cdots z}$$

This spin structure could be related to the Riemannian spin structure when antisymmetrization is replaced by symmetrization and n is replaced by $-n$. Thus, symplectic spinors are referred to as negative-dimensional spinors.

5. RELATIVITY STRUCTURE

5.1. Bergmann Manifold

An n-dimensional manifold is a set (the elements of which are called points) along with a topology (a guide to tell which points are near each other) such that any small region of points can be mapped 1–1 to an open set of $\mathbb{R}^n$. To build an n-dimensional spacetime, we need an n-dimensional manifold (called the base manifold) and a bracket space. The bracket defines a chronometric on the base manifold providing properties such as curvature and size. The group associated with the bracket is called the structure group. The manifold structure is chosen since it is more general than $\mathbb{R}^n$, yet retains the local $\mathbb{R}^n$ structure. We choose an N^2-dimensional manifold as the base manifold and the Bergmannian bracket space. Then the complex special linear group SL_N is the structure group, and the Bergmannian chronometric is the chronometric. Together these form a Bergmann manifold denoted B_N. We call the manifold "spacetime" and the points of the manifold "events of spacetime."

A fiber bundle is a manifold which is locally $M_1 \times M_2$ for some manifolds M_1 and M_2. If M_2 is a tangent space, then the fiber bundle is called a tangent bundle. At each event of B_N we shall place an N^2-dimensional tangent space, i.e., a linear vector space, obeying the Leibnitz rule. This describes a tangent bundle in which $M_1 = B_N$. Each special sesquispinor specifies a vector in the tangent space of some event of the spacetime. In order for the $N \times N$ complex special sesquispinor to be described by N^2 real variables, we require it to be Hermitian. Any tangent space can be described by an N-dimensional complex spin space which is equivalent (i.e., same number of variables) to an N^2-dimensional real vector space. The transformation properties are determined by applying the structure group to the tangent spaces of each event of the spacetime. We shall take the spin form $\sigma_a{}^{AA'}$ to be our field variable, just as the symmetric biform g_{ab} is taken to be the field variable of general relativity. We shall call $\sigma_a{}^{AA'}$ the metric form.

5.2. Assumptions of Relativity

We mean to create a theory which retains as much of the general relativity structure as possible. The following is a list of some principles which are assumed by the theory of general relativity.

1. The principle of general covariance.
2. Einstein's equivalence principle holds.
3. The stress-energy-momentum tensor T^{ab} is the source of the gravitational field g^{ab}.
4. There exists a four-dimensional, rank-two symmetric chronometric.
5. $T^c{}_{ab} = 0$ (spacetime is torsion-free).
6. $G^{ab} = \kappa T^{ab}$ (Einstein's equations hold).
7. The chronometric is constant under covariant differentiation.

We now state how these assumptions are changed by the use of a Bergmannian manifold B_N.

Assumption 1. Assumption 1 does not change.

Assumption 2. Local Lorentz invariance requires $SO(1, 3, \mathbb{R})$ symmetry. The orthogonal extension to an n-dimensional spacetime requires $SO(1, n, -1, \mathbb{R})$ symmetry. However, the group of a Bergmannian manifold B_N is $SL(N, \mathbb{C})$. Thus, local symmetry on B_N will be $SL(N, \mathbb{C})$ symmetry. Notice that the local Lorentz invariance will still hold for the four-dimensional spacetime.

Assumption 3. A tensor T^{ab}, with indices ranging from zero to $N^2 - 1$, will be the source of the gravitational field $g^{ab\cdots}$. The prefix "hyper" is added to the name when N is greater than two.

Assumption 4. In Assumption 4 the Einstein chronometric is replaced with an N^2-dimensional, rank-N Bergmannian chronometric. This is a special case of Finslerian geometry (Busemann, 1942).

Assumption 5. Although it is possible to find unique torsion-free connections (i.e., $\Gamma^c{}_{ab} = \Gamma^c{}_{ba}$) for Riemannian spacetimes, it is not always possible for $N > 2$ Bergmannian spacetimes (Borowiec, 1988). Connections with torsion have N^6 possible independent symbols, while those without torsion have $(N^6 + N^4)/2$. Since Assumption 5 cannot always hold, the connection cannot in general be expressed in terms of the chronometric and its derivatives. Thus, the action for spacetime has two independent dynamical variables $g_{abc\ldots}$ and $\Gamma^c{}_{ab}$.

Since $\Gamma^c{}_{ab}$ is not necessarily equal to $\Gamma^c{}_{ba}$, there are two independent differentials for vectors,

$$\nabla_b v^c = \partial_b v^c + \Gamma^c{}_{ab} v^a$$

or

$$\nabla_b v^c = \partial_b v^c + \Gamma^c{}_{ba} v^a$$

and, respectively, two independent differentials for covectors,

$$\nabla_b v_a = \partial_b v_a - \Gamma^c{}_{ab} v_c$$

or

$$\nabla_b v_a = \partial_b v_a - \Gamma^c{}_{ba} v_c$$

For the calculations in this paper, we have used the first definition in each of the two cases.

Assumptions 6 and 7. Assumptions 6 and 7 can be obtained in general relativity by varying an action. These assumptions are discussed in Section 5.4.

5.3. Raising Indices

The connection, torsion tensor, curvature tensor, and the (0, 2) Ricci tensor are not chronometric-dependent quantities. The torsion of spacetime is defined by $T(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y]$, and yields the torsion tensor

$$T^c{}_{ab} = \Gamma^c{}_{ab} - \Gamma^c{}_{ba}$$

in a geodesic coordinate basis (*gcb*). The curvature is defined by

$$R_{(X,Y)}Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and yields the curvature tensor

$$R^a{}_{bcd} = \Gamma^a{}_{db,c} - \Gamma^a{}_{cb,d} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}$$

The inverse Riemannian chronometric is defined by

$$g^{ab} = [\mathrm{co}(g_{ab})]/[\det(g_{ab})]$$

which implies $g^{ac}g_{bc} = \delta^a{}_b$. We require the inverse Bergmannian chronometric to obey

$$g^{acd\ldots} g_{bcd\ldots} = \delta^a{}_b$$

Thus, the $N > 2$ inverse Bergmannian chronometric is not as yet uniquely defined, except in special cases.

Let $\mathbf{\Gamma}^c{}_{ab}$ be the torsion-free connection (i.e., $\mathbf{\Gamma} = \Gamma|_{T=0}$) and let $K^c{}_{ab}$ be the contortion tensor, $K^c{}_{ab} \equiv (\Gamma^c{}_{ab} - \mathbf{\Gamma}^c{}_{ab})$. For the $N = 2$ case,

$$\mathbf{\Gamma}^c{}_{ab} = \tfrac{1}{2} g^{cd}(g_{da,b} + g_{db,a} - g_{ab,d})$$

$$K^c{}_{ab} = \tfrac{1}{2} g^{cd}(g_{ea}T^e{}_{dc} + g_{eb}T^e{}_{da}) - \tfrac{1}{2} T^c{}_{ba}$$

which may be derived using equation (4) of Appendix A.1. Notice that for $N = 2$ the symmetric part of the connection, $(\Gamma^c{}_{ab} + \Gamma^c{}_{ba})/2$, is not torsion-free. The (0, 2) Ricci tensor is defined as $R_{ab} \equiv R^c{}_{acb}$. The (1, 1) Ricci tensor $R^a{}_b$ and the Ricci scalar $R \equiv R^a{}_a$ depend on the (0, 2) Ricci tensor and a tensor which raises indices.

A two-index tensor which reduces to the chronometric for $N = 2$ is required in order to raise and lower tensor indices. In a Riemannian spacetime the chronometric fulfills this requirement. There are a few possibilities for this tensor in a Bergmannian space, all of which reduce to the chronometric for $N = 2$.

1. The Ricci tensor is of the necessary rank, thus suggesting setting g_{ab} proportional to R_{ab} for the $N = 2$ case. In this case $\nabla g_{ab} \neq 0$.

2. Often Finsler metrics are contracted with vectors to lower the tensorial rank. Perhaps the Bergmannian chronometric along with appropriate contractions could be used.

3. Using covariant differentiation to contract with all but two of the metric indices, $T^a = g^{abc\ldots} T_{b;c\ldots}$. In this case $\nabla g_{ab} \neq 0$.

5.4. Varying the Action

In general relativity $\rho = [-\det(g_{ab})]^{1/2}$ is used to compensate for the Jacobian in the action. Since $\sigma \sim (g)^{1/2}$, we use $\rho = \det[(\sigma^a{}_\alpha)^{-1}]$, where $\alpha = AA'$, called the chronometric density (note: ρ is not a tensor). The (1, 1) Ricci tensor shall be defined by $R^a{}_b \equiv g^{acd\ldots} R_{bc;d\ldots}$ and the action by

$$S \equiv \rho R \equiv \rho g^{abc\ldots} R_{ab;c\ldots}$$

Using

$$\delta\rho = -\rho\sigma_d{}^{AA'} \delta\sigma^d{}_{AA'} = -\rho\sigma^{-1}\delta\sigma$$

$$\delta g^{abc\ldots} = N\sigma^b{}_{BB'}\sigma^c{}_{CC'\ldots}\varepsilon^{ABC\ldots}\varepsilon^{A'B'C'\ldots}\delta\sigma^a{}_{AA'}$$

$$\delta[R_{ab;c\ldots}] = [\delta\Gamma^\alpha{}_{ab}]_{;\alpha c\ldots} - [\delta\Gamma^\alpha{}_{a\alpha}]_{;bc\ldots}$$

and setting $\delta S = 0$, we find

$$\int \rho\{[-R\sigma^{-1} + N\mu]\delta\sigma + g^{abc\ldots}\delta[R_{ab;c\ldots}]\}\, d^n x = 0$$

where

$$\mu_a{}^{AA'} = \sigma^b{}_{BB'}\sigma^c{}_{CC'\ldots}\varepsilon^{ABC\ldots}\varepsilon^{A'B'C'\ldots}R_{ab;c\ldots}$$

and where R is the Ricci scalar. Since $R_{ab;c\ldots}$ is a function of the connection alone, and (σ, Γ) are independent variables, we obtain the following two variation equations:

$$\int \rho[-R\sigma^{-1} + N\mu]\, \delta\sigma\, d^n x = 0$$

$$\int \rho g^{abc\ldots}\, \delta[R_{ab;c\ldots}]\, d^n x = 0$$

Assumption 6. By contracting the first variation equation with $\sigma^d{}_{AA'}$, we find

$$R^a{}_b - \frac{1}{N}\delta^a{}_b R = 0$$

which shall be called the vacuum hyper-Einstein equations (see Appendix A.3). This equation reduces to the vacuum field equations of general relativity for $N = 2$.

For systems with mass, introduce the hyper-stress-energy-momentum tensor $T^a{}_b$, such that

$$T^a{}_b = [\sigma^a{}_{AA'}\,\delta(\rho L_M)]/[\kappa N \rho\, \delta\sigma^b{}_{AA'}]$$

where L_M is a function of the matter field ϕ and the chronometric. If we assume the action is additive, i.e.,

$$S = \rho R + \rho L_M$$

then the first variation equation becomes

$$R^a{}_b - \frac{1}{N}\,\delta^a{}_b R = \kappa T^a{}_b$$

By contracting this equation, we find $\rho = -\kappa T/(N-1)$, where $T = T^a{}_a$ (note: $\delta^a{}_a = N^2$). Thus,

$$R^a{}_b = \kappa\left[T^a{}_b - \frac{\delta^a{}_b T}{N^2 - N}\right]$$

The Newtonian limit is obtained by setting $v \ll c$ and neglecting nonlinear terms in the connection. The following equations are a direct result of the Newtonian limit (see Appendices A.4–A.6 for derivations in the Newtonian limit):

1. $c^2\,dt^2 \gg c\,dt\,dx \gg dx\,dy$
2. $d^2x^k/dt^2 = -c^2\Gamma^k{}_{00}$
3. $R_{00} = \Gamma^k{}_{00,k}$
4. $T = T^0{}_0$
5. $R^0{}_0 = g^{00b\ldots}\Gamma^a{}_{00,ab\ldots}$

Here $k = 1, \ldots, n-1$ and $i = 4, \ldots, n-1$. From these five equations, we obtain

$$g^{00b\ldots}\Gamma^a{}_{00,ab\ldots} = -\kappa T^0{}_0(N^2 - N - 1)/(N^2 - N)$$

Let ρ be the mass density and set $T_{00} = c^2\rho$. Assuming (1)

$$T^0{}_0 = c^2 g^{00a\ldots}\rho_{;a\ldots}$$

and (2) there exists a function ϕ such that

$$\Gamma^a{}_{00} = -(\partial\phi/\partial x_a)/c^2$$

i.e., there exists a conservative gravitational field, then

$$g^{00a\ldots b}\{\nabla^2{}_{n-1}\phi + \kappa[(N^2 - N - 1)/(N^2 - N)]c^2\rho\}_{;a\ldots b} = 0$$

Also assume that

$$\nabla^2_{n-1}(\phi) = \Omega_{n-1} G\rho$$

where Ω_{n-1} is the solid angle in $(n-1)$-dimensional space ($\Omega_3 = 4\pi$). Then we find

$$\kappa = \frac{N^2 - N}{N^2 - N - 1} \Omega_{N^2-1} \frac{G}{c^2}$$

This result agrees exactly with that of general relativity for $N=2$. We then replace Assumption 6 of Section 5.2 with

$$R^a{}_b - \frac{1}{N} \delta^a{}_b R = \frac{N^2 - N}{N^2 - N - 1} \Omega_{N^2-1} \frac{G}{c^2} T^a{}_b$$

Assumption 7. The second variation equation is

$$\int \rho g^{abc\ldots} \, \delta[R_{ab;c\ldots}] \, d^n x = 0$$

Define the tensor $D_{abc\ldots}$ by

$$\delta(R_{ab;c\ldots}) = (\delta R_{ab})_{;c\ldots} + D_{abc\ldots}$$

Note the following:

$$(\delta R_{ab}) - \delta(R_{ab}) = 0 \qquad (N=2)$$

$$(\delta R_{ab})_{;c} - \delta(R_{ab;c}) = (\delta \Gamma^{\alpha}{}_{ca}) R_{ab} + (\delta \Gamma^{\alpha}{}_{cb}) R_{a\alpha} \qquad (N=3)$$

$$(\delta R_{ab})_{;cd} - \delta(R_{ab;cd}) = (\delta \Gamma^{e}{}_{dc}) R_{ab;e} - D_{abc;d} - D_{abd;c} \qquad (N=4)$$

$$(\delta R_{ab})_{;cd} \neq \delta(R_{ab;cd}) \qquad (N>2)$$

Then using the results of Appendices A.7 and A.8, we can write the second variation equation as

$$(-1)^{N-1} \int [(\rho g^{abc\ldots})_{;\ldots ca} - (\rho g^{a\beta c\ldots})_{;\ldots c\beta} \delta^b{}_a] \, \delta \Gamma^{\alpha}{}_{ab} \, d^n x$$

$$+ \int \rho g^{abc\ldots} D_{abc\ldots} \, d^n x = 0$$

The quantity ρ is neither a scalar nor a tensor, it is a density. There is no natural covariant derivative defined for densities. We define $\rho_{;a} = 0$. Thus, the second variation equation can be written as,

$$(-1)^{N-1}\int [\rho(g^{abc\ldots})_{;\ldots ca}-\rho(g^{a\beta c\ldots})_{;\ldots c\beta}\delta^{b}{}_{\alpha}]\,\delta\Gamma^{a}{}_{ab}\,d^{n}x$$
$$+\int \rho g^{abc\ldots}D_{abc\ldots}\,d^{n}x=0$$

For $N=2$, the second variation equation becomes (see Appendix A.9)

$$[(g^{ab})_{;\alpha}-(g^{a\beta})_{;\beta}\delta^{b}{}_{\alpha}]=0$$

or equivalently,

$$g^{ab}{}_{;c}=0$$

Also this equation yields

$$g_{ab;c}=0$$

For $N=3$, the second variation equation becomes (see Appendix A.10),

$$[(g^{abc})_{;c\alpha}-(g^{a\beta c})_{;c\beta}\delta^{b}{}_{\alpha}-g^{abc}(R_{\alpha c}+R_{c\alpha})]=0$$

Summing over b and α yields

$$(g^{abc})_{;cb}=-g^{abc}R_{bc}/4$$

where the 4 comes from $(n-1)/2$. The constraint $\nabla g=0$ for a Riemannian chronometric falls out as a result of the second variation equation. This does not occur for an $N>2$ Bergmannian chronometric. If we assume $g^{abc}{}_{;c}=0$, then

$$g^{abc}(R_{ac}+R_{ca})=0$$

and

$$g^{abc}R_{bc}=0$$

Thus, Assumption 7 shall be unchanged for $N=2$, but quite different for $N>2$.

This concludes the changes made to the assumptions of general relativity.

6. "SPHERICALLY" SYMMETRIC BERGMANNIAN CHRONOMETRIC

6.1. SU_N Invariants

The set of $N\times N$ unitary matrices with determinant one is a representation of the special unitary group [often denoted $SU(N,\mathbb{C})$ or SU_N], which has N^2-1 parameters. SL_N can be decomposed into spatial rotations, represented by the group SU_N, and boosts, represented by the space $\mathbb{R}^{N^2-1}$, once a time axis is chosen.

If $x=T(x)$, then x is said to be invariant under that transformation. If an object is invariant under all transformations T defined by elements of a group, then the object is said to be invariant under that group. For example, the set of 3×3 orthogonal matrices with respect to the identity represent the rotation group denoted $O(3, 0)$ or O_3. When a vector is transformed under O_3, that vector undergoes a rotation in the three-dimensional space. The magnitude of a vector dr given by $dr^2=dx^2+dy^2+dz^2$ does not change under rotation. Thus, r is said to be O_3-invariant.

Just as one might change from rectangular coordinates to spherical coordinates, we will change from the N complex special sesquispinor coordinates to the N^2 real coordinates, such that some of the variables are SU_N-invariant (such as the time and radius coordinates of orthogonal spacetime). The special sesquispinor contains a complete description of an event in the Bergmann spacetime and can be represented as a matrix, so changing coordinate systems can be accomplished through similarity transformations of the special sesquispinor. The trace of a matrix is always invariant under similarity transformations; therefore the trace is SU_N-invariant. We choose

$$t=\langle X\rangle \equiv \frac{1}{N}(\text{trace } X)$$

to be the time coordinate. Since this is the only linear invariant, we will not include any other time dimensions in this theory. Thus, no special assumptions are necessary for local causality, i.e., Bergmann spacetimes are inherently causal for all N. Let the spatial spin matrix be defined as

$$X_S=X-\langle X\rangle\sigma_0$$

X_S describes an (N^2-1)-dimensional manifold which we shall call space. The determinant of a matrix is also invariant under similarity transformations; therefore other SU_N-invariants can be found through the characteristic polynomial of the spatial spin matrix; $[\det(X_S-\lambda\sigma_0)=0]$. The solution to this equation yields $N-1$ independent SU_N-invariants of degree 2 through N, respectively, given by

$$r_m \equiv (-1)^{N-m+1}\left[\frac{\partial}{\partial\lambda}\right]^{N-m}[\det(X_S-\lambda\sigma_0)]\,|_{\lambda=0}$$

for $m=2, 3, \ldots, N$. The minus sign is chosen such that r_2 will be positive for all N. Then r_m is the mth-order SU_N-invariant with $r_1=t$.

The orthogonal group O_n has one invariant, $r=(x_1{}^2+\cdots+x_n{}^2)$. The unitary group SU_N has $N-1$ invariants, $r_2, r_3, \ldots, r_N$. The O_n-invariant r is of order 2, while the SU_N-invariants r_m are of order m. So for spacetime dimensions $n>4$ ($n=N^2$) there are more SU_N-invariants than O_n-invariants. These SU_N-invariants can be written as

$$r_N=\pm\frac{1}{N!}\,\varepsilon_{A_1\ldots A_N}\varepsilon_{A_1'\ldots A_N'}X_S^{A_1A_1'}\cdots X_S^{A_NA_N'}$$

$$r_{N-1}=\pm\frac{1}{(N-1)!}\,\varepsilon_{A_1\ldots A_N}\varepsilon_{A_1'\ldots A_N'}\delta^{A_1A_1'}X_S^{A_2A_2'}\cdots X_S^{A_NA_N'}$$

$$\vdots$$

$$r_2=\pm\tfrac{1}{2}\,\varepsilon_{A_1\ldots A_N}\varepsilon_{A_1'\ldots A_N'}\delta^{A_1A_1'}\cdots\delta^{A_{N-2}A_N'-2}X_S^{A_{N-1}A_N'-1}\cdots X_S^{A_NA_N'}$$

$$t=\pm\frac{1}{N}\,\delta_{AB}X^{AB}$$

where the delta symbols δ_{AB} and δ^{AB} are one if $A=B$ and zero otherwise and where the N-dimensional Levi-Civita symbol $\varepsilon_{AB\ldots C}$ is zero if there are repeated indices, one for $\varepsilon_{12\ldots N}$ and changes sign under index permutation.

6.2. Omega Space

There are N SU_N-invariants, so we need N^2-N angles to complete the set of coordinates for the N^2-dimensional hyperspace. We shall call the space of angles the omega space. A method of determining these angles requires the group theory concepts of coset, normal subgroup, and factor group.

If S is a subgroup of the group G, then a left coset is defined by $\{gs\,|\,s\in S\}$ and a right coset by $\{sg\,|\,s\in S\}$ for some element g of G. Each element of G defines a left and a right coset of S in G. Two left (right) cosets are either identical or have no elements in common. Therefore a complete set of left (right) cosets can be obtained by using all elements of G to define cosets and removing any repeats. This complete set of left (right) cosets partitions G.

S in a normal subgroup of G if for all g in G and for all s in S, $(gsg^{-1})\in S$. As an example, let G be a group represented by matrices, and let $[G]$ denote the subgroup of G consisting of diagonal matrices in G. Since diagonal matrices commute with all matrices, $[G]$ is a normal subgroup of G. For all normal subgroups the left coset and right coset generated by g are the same.

A factor group (denoted G/S) consists of elements which are cosets of S in G, where S is a normal subgroup of G. This group is also called the quotient group. $G/[G]$ is an example of a factor group.

An element of $[SU_N]$ can be written as the matrix

$$\operatorname{diag}[\exp(-i\alpha_1/2), \ldots, \exp(-i\alpha_{N-1}/2), \exp(i(\alpha_1 + \cdots + \alpha_{N-1})/2)]$$

These diagonal elements are called the eigenvalues. Define

$$G^N \equiv SU_N/[SU_N]$$

and call this group the globe of SU_N. The number of parameters for G^N is (number of parameters in SU_N) − (number of parameters in $[SU_N]$) = $(N^2-1)-(N-1)=N^2-N$, which is the number of angles required for the omega space. We set G^N to represent this space.

The spatial spin matrix X_S may have degeneracies among the eigenvalues (also recall X_S is traceless). The case where there are no degeneracies is called regular. Singularities in the spacetime metric fall into two categories, coordinate-dependent singularities and true singularities. The Earth's geographic poles and the Schwarzschild radius $r=2M$ are examples of coordinate-dependent singularities, since observers passing these points would see no physical significance to these points. Singularities do not occur in the regular parts of spacetime unless there is a black hole present. Thus, to find black holes, first search the regular parts of spacetime for singularities.

We partition G_N into the following classes of degeneracies:

$N=2$	* *	$r_2=0$	(origin)
	/	$r_2>0$	(regular)
$N=3$	* * *	$r_3=r_2=0$	(origin)
	* */*	$r_3^2=\alpha r_2^3\neq 0$	
	//*	$r_3^2\neq\alpha r_2^3$	(regular)
$N=4$	* * * *	$r_4=r_3=r_2=0$	(origin)
	* * */*	$r_4^3=\beta r_3^4=\gamma r_2^6\neq 0$	
	* */* *	$r_4=\delta r_2^2\neq 0,\ r_3=0$	
	* */*/*	Mess	
	//*/*	All else	(regular)

where

$$\alpha=\frac{2^2}{3^3},\qquad \beta=\frac{3^3}{10^4},\qquad \gamma=\frac{3^3}{6^6},\qquad \delta=-\frac{1}{4}$$

The fully degenerate case is called the origin, while the fully nondegenerate case is called the regular portion of G^N. There are P_N classes for each spinor dimension, where P_N is the number of partitions of N. Each class, called an orbit, contains all events of spacetime which can be rotated into each other.

The intersection of any two classes is the null set and the union of all the classes is G_N. The fully degenerate class has but one event, the origin. The $N=3$ and $N=4$ results above suggest that the regular parts of spacetime can be approximated as the union of solid cones.

6.3. Two-Spinor Example

The following examples illustrate the previous concepts for the case of $N=2$.

A set of constant spin matrices called the Pauli spin matrices are given by

$$\mathring{\sigma}_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \mathring{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \mathring{\sigma}_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \mathring{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

A special sesquispinor can be described by

$$X_2=t\mathring{\sigma}_0+x\mathring{\sigma}_1+y\mathring{\sigma}_2+z\mathring{\sigma}_3=\begin{pmatrix}t+z & x-iy\\ x+iy & t-z\end{pmatrix}$$

Then the characteristic polynomial of the spatial spin matrix is given by

$$\det[X_2-(t+\lambda)\mathring{\sigma}_0]=\lambda^2-r_2$$

and the SU_2-invariants are then

$$\langle X\rangle=t$$
$$r_2=x^2+y^2+z^2$$

An arbitrary element of SU_2 can be written as

$$U(\alpha,\beta,\gamma)=\begin{pmatrix}e^{i(\gamma+\alpha)/2} & \cos\beta/2 & e^{i(\gamma-\alpha)/2} & \sin\beta/2\\ -e^{-i(\gamma-\alpha)/2} & \sin\beta/2 & e^{-i(\gamma+\alpha)/2} & \cos\beta/2\end{pmatrix}$$

An arbitrary element of G^2 can be written as

$$Q(\beta,\gamma)=\begin{pmatrix}e^{-i\alpha/2} & 0\\ 0 & e^{i\alpha/2}\end{pmatrix}U(\alpha,\beta,\gamma)=\begin{pmatrix}e^{i\gamma/2} & 0\\ 0 & e^{-i\gamma/2}\end{pmatrix}\begin{pmatrix}\cos\beta/2 & \sin\beta/2\\ -\sin\beta/2 & \cos\beta/2\end{pmatrix}$$

The spacetime variables for the "spherically" symmetric spacetime are then given by (t, r_2, Ω), where $\Omega=(\gamma,\beta)$.

6.4. Three-Spinor Example

The following examples illustrate the previous concepts for the case of $N=3$.

A set of extended spin matrices is given by

$$\mathring{\sigma}_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad \mathring{\sigma}_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\qquad \mathring{\sigma}_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},$$

$$\mathring{\sigma}_3=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\qquad \mathring{\sigma}_4=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},$$

$$\mathring{\sigma}_5=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\qquad \mathring{\sigma}_6=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},$$

$$\mathring{\sigma}_7=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\qquad \mathring{\sigma}_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

A special sesquispinor can be described by

$$X_3=t\mathring{\sigma}_0+x_1\mathring{\sigma}_1+x_2\mathring{\sigma}_2+y_1\mathring{\sigma}_3+y_2\mathring{\sigma}_4+y_3\mathring{\sigma}_5+y_4\mathring{\sigma}_6+z_1\mathring{\sigma}_7+z_2\mathring{\sigma}_8$$

Then the characteristic polynomial of the spatial spin matrix is given by

$$\det[X_3-(t+\lambda)\mathring{\sigma}_0]=-\lambda^3+r_2\lambda-r_3$$

and the SU_3-invariants are

$$\langle X\rangle=t$$

$$\begin{aligned} r_3&=z_2[(y_1^2+y_2^2+y_3^2+y_4^2)-2(x_1^2+x_2^2+z_1^2+z_2^2)]+z_1[y_1^2+y_2^2-y_3^2-y_4^2]\\ &\quad+2[x_1(y_1y_3+y_2y_4)+x_2(y_2y_3-y_1y_4)]\\ r_2&=x_1^2+x_2^2+y_1^2+y_2^2+y_3^2+y_4^2+z_1^2+z_2^2 \end{aligned}$$

An arbitrary element of SU_3 can be written as

$$U(\alpha_1,\ldots,\alpha_8)=Q(\alpha_8)Q(\alpha_7)\cdots Q(\alpha_1)$$

where

$$Q(\alpha_i)=\mathbf{1}\cos(\alpha_i/2)+\sigma_i\sin(\alpha_i/2)$$

An arbitrary element of G^3 can be written as

$$Q(\alpha_1,\ldots,\alpha_6)=Q^{\dagger}(\alpha_7)Q^{\dagger}(\alpha_8)U(\alpha_1,\ldots,\alpha_8)$$

The spacetime variables for the "spherically" symmetric spacetime are (t, r_2, r_3, Ω), where $\Omega=(\alpha_1,\ldots,\alpha_6)$.

6.5. Chronometric Constraints

The time translation operator is $T_t = (x \to x + \Delta t\delta)$. The time reversal operator is $T_r = (t \to -t)$. The invariant for Bergmann spaces is

$$d\tau^N = g_{abc\ldots}\, dx^a\, dx^b\, dx^c \ldots$$

We shall require g to be invariant under T_t and T_r. T_t requires that g be independent of time and T_r requires that dt appear in the volume element with only even powers for N even, or either odd powers or even powers for N odd (depending on the required sign change of $d\tau^N$). The only linear scalar that can be constructed with the omega space is $(d\Omega)^2$, so no other powers of $d\Omega$ shall appear in the volume element.

For $N=2$, these requirements (along with Einstein's equations) yield the Schwarzschild metric. For $N=3$ they yield

$$\begin{aligned} d\tau^3 &= dt\,(M\,dt^2 + N_{\mu\nu}\,dx^\mu\,dx^\nu); \qquad \mu, \nu = 1, \ldots, 8 \\ &= dt\,(e^\omega\,dt^2 - e^\lambda\,dr^2 - e^\lambda\,dp^2 - R^2\,d\Omega^2) \\ &= dt\,(e^\omega\,dt^2 - e^\lambda\,dr^2 - e^\lambda\,dp^2 - R^2 f_{\alpha\beta}\,d\theta^\alpha\,d\theta^\beta) \end{aligned} \tag{1}$$

where $f_{\alpha\beta} = f_{\beta\alpha}$ and $\alpha, \beta = 3, \ldots, 8$; or

$$\begin{aligned} d\tau^3 &= dx^\mu(\tfrac{1}{3}g_{\mu 00}\,dt^2 + g_{\mu\nu\gamma}\,dx^\nu\,dx^\gamma); \qquad \mu, \nu, \gamma = 1, \ldots, 8 \\ &= (A_1\,dr + B_1\,dq)\,dt^2 - (A_2\,dr + B_2\,dq)(e^\lambda\,dr^2 + e^\nu\,dq^2) \\ &\quad (A_3\,dr + B_3\,dq)\,d\Omega^2 \end{aligned} \tag{2}$$

where R, ω, λ, A_i, and B_i are functions of $r = (r_2)^{1/2}$ and $p = (r_3)^{1/3}$.

To find flat spacetime, let

$$d\Omega = \omega = \lambda = 0$$

$$A_i = B_i = 1$$

and let t return to ct in the volume element equation:

1. $d\tau^3 = c\,dt\,[c^2(dt)^2 - \sum (dx_i)^2]$
2. $d\tau^3 = (dr + dq)[c^2(dt)^2 - \sum (dx_i)^2]$

To find the light cones, set $d\tau$ equal to zero.

1. $N=2$:

$$\frac{dr}{dt} = \pm c \qquad \text{(usual light cone)}$$

2. $N=3$, formula 1:

$$\frac{dr}{dt}=\pm\left[\left(\frac{dp}{dt}\right)^2+c^2\right]^{1/2}$$

$$dt=0$$

3. $N=3$, formula 2:

$$\frac{dr}{dt}=\pm\left[\left(\frac{dp}{dt}\right)^2+c^2\right]^{1/2}$$

$$\frac{dr}{dt}=-\frac{dp}{dt}$$

Setting dp/dt equal to zero yields interesting results. Then for $N=3$ the first equation for each formula is equivalent to the usual light cone. For formula 1, the new light cone restriction, $dt=0$, will separate the past from the present, which does not conflict with physical phenomenon. For formula 2, the new light cone restriction, $dr/dt=0$, is not as satisfying.

For $N=3$ let:

1. $r=(r_1)^{1/2}$, $p=(r_3)^{1/3}$.
2. R, ω, λ are functions of r, p only.
3. $f_{\alpha\beta}=f_{\beta\alpha}$; α, $\beta=1,\ldots,6$.
4. $dp/dt=0$.

In conclusion, we suggest the nine-dimensional spacetime interval

$$d\tau^3=dt\,(e^{\omega}\,dt^2-e^{\lambda}\,dr^2-e^{\lambda}\,dp^2-R^2f_{\alpha\beta}\,d\theta^{\alpha}\,d\theta^{\beta})$$

as a "spherical" symmetry analog to the four-dimensional Schwarzschild invariant.

APPENDIX A. DERIVATIONS FOR SECTION 5

A.1

Assume

$$g_{ab;c}=0$$

Then

$$g_{bc;a}=0$$

and

$$g_{ca;b}=0$$

So

$$g_{ab,c}-g_{db}\Gamma^{d}{}_{ca}-g_{ad}\Gamma^{d}{}_{cb}=0 \tag{A1}$$

$$g_{bc,a}-g_{dc}\Gamma^{d}{}_{ab}-g_{bd}\Gamma^{d}{}_{ac}=0 \tag{A2}$$

$$g_{ca,b}-g_{da}\Gamma^{d}{}_{bc}-g_{cd}\Gamma^{d}{}_{ba}=0 \tag{A3}$$

Combining (A3) + (A2) − (A1) = 0 yields

$$\begin{aligned} &g_{ca,b}+g_{bc,a}-g_{ab,c}-g_{da}(\Gamma^{d}{}_{bc}-\Gamma^{d}{}_{cb}) \\ &\quad -g_{db}(\Gamma^{d}{}_{ac}-\Gamma^{d}{}_{ca})-g_{dc}(\Gamma^{d}{}_{ab}+\Gamma^{d}{}_{ba})=0 \end{aligned}$$

Recall that $T^{d}{}_{ab}$, the torsion tensor, is given by

$$\Gamma^{d}{}_{ab}=\Gamma^{d}{}_{ba}+T^{d}{}_{ab}$$

Then,

$$\begin{aligned} &g_{ca,b}+g_{bc,a}-g_{ab,c}-g_{da}T^{d}{}_{bc}-g_{db}T^{d}{}_{ac}-g_{dc}([\Gamma^{d}{}_{ab}+\Gamma^{d}{}_{ba})=0 \\ &g_{dc}\Gamma^{d}{}_{ab}+g_{dc}\Gamma^{d}{}_{ba} \\ &\quad =(g_{ca,b}+g_{bc,a}-g_{ab,c})+(g_{da}T^{d}{}_{cb}+g_{db}T^{d}{}_{ac}) \end{aligned} \tag{A4}$$

$$[ab, c]+[ba, c]+\hat{C}_{abc}+\hat{T}_{abc}$$

Let $T^{c}{}_{ab}=0$ (torsion-free). Then $\hat{T}_{abc}=0$. So

$$[ab, c]+[ba, c]=\hat{C}_{abc}$$

A.2

Assume

$$g_{abc;d}=0$$

Then

$$g_{bcd;a}=0$$

$$g_{cda;b}=0$$

$$g_{dab;c}=0$$

So

$$g_{abc,d} - g_{ebc}\Gamma^e{}_{da} - g_{aec}\Gamma^e{}_{db} - g_{abe}\Gamma^e{}_{dc} = 0 \tag{A5}$$

$$g_{bcd,a} - g_{ecd}\Gamma^e{}_{ab} - g_{bed}\Gamma^e{}_{ac} - g_{bce}\Gamma^e{}_{ad} = 0 \tag{A6}$$

$$g_{cda,b} - g_{eda}\Gamma^e{}_{bc} - g_{cea}\Gamma^e{}_{bd} - g_{cde}\Gamma^e{}_{ba} = 0 \tag{A7}$$

$$g_{dab,c} - g_{eab}\Gamma^e{}_{cd} - g_{deb}\Gamma^e{}_{ca} - g_{dae}\Gamma^e{}_{cb} = 0 \tag{A8}$$

Combining $[(A5)+(A6)]-[(A7)+(A8)]=0$, and following the same procedure as that of Appendix A.3, we obtain

$$[ad, bc] + [bc, ad] = \hat{C}_{abcd} - \hat{T}_{abcd}$$

Let $T_{ab}=0$ (torsion-free). Then $\hat{T}_{abcd}=0$. So

$$[ad, bc] + [bc, ad] = \hat{C}_{abcd}$$

A.3

Let

$$S = \rho g^{abc\dots} R_{ab;c\dots}$$

Then

$$\begin{aligned}\delta S = {} & [-\rho\sigma_a{}^{AA'}\delta\sigma^a{}_{AA'}] g^{abc\dots} R_{ab;c\dots} \\ & + \rho R_{ab;c\dots}[N\sigma^b{}_{BB'}\sigma^c{}_{CC'} \cdots \varepsilon^{ABC\dots}\varepsilon^{A'B'C'\dots} \cdots \delta\sigma^a{}_{AA'}] \\ & + f(\delta\Gamma)\end{aligned}$$

$$\begin{aligned}\frac{\delta S}{\delta\sigma^a{}_{AA'}} = {} & [-\rho\sigma_a{}^{AA'}R] \\ & + N\rho R_{ab;c\dots}\sigma^b{}_{BB'}\sigma^c{}_{CC'} \cdots \varepsilon^{ABC\dots}\varepsilon^{A'B'C'\dots} \cdots\end{aligned}$$

Set the above equal to zero, and contract with $\sigma^a{}_{AA'}$:

$$[\sigma^a{}_{AA'}\sigma_a{}^{AA'}R + R_{ab;c\dots}\sigma^a{}_{AA'}\sigma^b{}_{BB'}\sigma^c{}_{CC'} \cdots \varepsilon^{ABC\dots}\varepsilon^{A'B'C'\dots} \cdots] = 0$$

$$-\delta^a{}_a R + N R_{ab;c\dots} g^{abc\dots} = 0$$

$$-\delta^a{}_a R + N R^a{}_a = 0$$

$$R^a{}_b - \frac{1}{N}\delta^a{}_b R = 0$$

A.4

We have

$$R_{ab} - \Gamma^{c}{}_{ba,c} - \Gamma^{c}{}_{ca,b} + \Gamma^{c}{}_{cd}\Gamma^{d}{}_{ba} - \Gamma^{c}{}_{bd}\Gamma^{d}{}_{ca}$$

For the Newtonian limit, products of Γ vanish,

$$R_{ab} = \Gamma^{c}{}_{ba,c} - \Gamma^{c}{}_{ca,b}$$

Then

$$R_{00} = \Gamma^{c}{}_{00,c} - \Gamma^{c}{}_{c0,0}$$

For the Newtonian limit, time derivatives vanish,

$$R_{00} = \Gamma^{c}{}_{00,c}$$

Since we have set

$$R^{a}{}_{b} = g^{abc\ldots} R_{ab;c\ldots}$$

we obtain

$$R^{0}{}_{0} = g^{00c\ldots} R_{00;c\ldots}$$

and

$$R^{0}{}_{0} = g^{00b\ldots}[\Gamma^{a}{}_{00,a}]_{;b\ldots}$$

A.5

Recall,

$$R^{a}{}_{b} = \kappa[T^{a}{}_{b} - \delta^{a}{}_{b}T/(N^2 - N)]$$
$$R^{0}{}_{0} = \kappa[T^{0}{}_{0} - T/(N^2 - N)]$$

For the Newtonian limit, $T = T^{0}{}_{0}$. We have

$$R^{0}{}_{0} = \kappa T^{0}{}_{0}[1 - 1/(N^2 - N)]$$
$$R^{0}{}_{0} = \kappa T^{0}{}_{0}[(N^2 - N - 1)/(N^2 - N)]$$

From Appendix A.4

$$R^{0}{}_{0} = g^{00b\ldots}[\Gamma^{a}{}_{00,a}]_{;b\ldots}$$

Thus,

$$g^{00b\ldots}[\Gamma^{a}{}_{00,a}]_{;b\ldots} = \kappa T^{0}{}_{0}[(N^2 - N - 1)/(N^2 - N)]$$

A.6

Let

$$\Gamma^{a}{}_{00} = -(\partial\phi/\partial x_a)/c^2$$

Then,

$$\begin{aligned} -c^2[\Gamma^{a}{}_{00,a}]_{;b\ldots} &= [\partial_a(\partial^a\phi)]_{;b\ldots} \\ &= [\partial_a\partial^a\phi]_{;b\ldots} \\ &= [\nabla^2_{n-1}(\phi)]_{;b\ldots} \end{aligned}$$

Assuming $T^0{}_0 = c^2 g^{00b\ldots}\rho_{;b\ldots}$, and using Appendix A.5, we find

$$g^{00a\ldots}\{\nabla^2_{n-1}(\phi) + \kappa[(N^2-N-1)/(N^2-N)]c^4\rho\}_{;a\ldots} = 0$$

We set

$$\nabla^2_{n-1}(\phi) = -\kappa[(N^2-N-1)/(N^2-N)]c^4\rho$$

A.7

We have

$$R_{ab} = [\Gamma^{\alpha}{}_{ab,\alpha} - \Gamma^{\alpha}{}_{a\alpha,b} + \Gamma^{\beta}{}_{ab}\Gamma^{\alpha}{}_{\beta\alpha} - \Gamma^{\beta}{}_{a\alpha}\Gamma^{\alpha}{}_{\beta b}]$$

In a local coordinate basis,

$$\delta R_{ab} = \delta[\Gamma^{\alpha}{}_{ab,\alpha}] - \delta[\Gamma^{\alpha}{}_{a\alpha,b}]$$

Since δ commutes with partial derivatives,

$$\delta R_{ab} = [\delta\Gamma^{\alpha}{}_{ab}]_{,\alpha} - [\delta\Gamma^{\alpha}{}_{a\alpha}]_{,b}$$

Since we are in a local coordinate basis,

$$\delta R_{ab} = [\delta\Gamma^{\alpha}{}_{ab}]_{;\alpha} - [\delta\Gamma^{\alpha}{}_{a\alpha}]_{;b}$$

This equation is tensorial, it is true in any coordinate basis. Note that this equation is true for any dimension N. For $N=2$, this equation is the Palatini identity. Thus, for $N>2$, we shall call this equation the hyper-Palatini identity.

A.8

We have

$$I=\int \rho g^{abc\ldots}[\delta R_{ab}]_{;c\ldots}\, d^n x$$

$$=\int \rho g^{abc\ldots}[(\delta\Gamma^{\alpha}{}_{ab;\alpha}-\delta\Gamma^{\alpha}{}_{a\alpha;b})_{;c\ldots}]\, d^n x$$

Integrate by parts $N-2$ times (boundary times vanish)

$$I=(-1)^{N-2}\int [\rho g^{abc\ldots}]_{;\ldots c}[(\delta\Gamma^{\alpha}{}_{ab;\alpha}-\delta\Gamma^{\alpha}{}_{a\alpha;b})]\, d^n x$$

Integrate by parts once more for each term,

$$I=(-1)^{N-1}\int [(\rho g^{abc\ldots})_{;\ldots c\alpha}-(\rho g^{a\beta c\ldots})_{;\ldots c\beta}\delta^{b}{}_{\alpha}]\, \delta\Gamma^{\alpha}{}_{ab}\, d^n x$$

A.9

For $N=2$,

$$D_{ab}=0$$

So the second variation equation is

$$\int \rho[(g^{ab})_{;c}-(g^{ad})_{;d}\delta^{b}{}_{c}]\, \delta\Gamma^{c}{}_{ab}\, d^4 x=0$$

Thus, set

$$[(g^{ab})_{;c}-(g^{ad})_{;d}\delta^{b}{}_{c}]=0$$

Recalling $g^{ab}=g^{ba}$, the only solution to this equation is

$$g^{ab}{}_{;c}=0$$

By lowering the indicies via the chronometric, we also find

$$g_{ab;c}=0$$

A.10

For $N=3$,

$$D_{abc}=(\delta\Gamma^{d}{}_{ac})R_{db}+(\delta\Gamma^{d}{}_{bc})R_{ad}$$

So the second variation equation is

$$(-1)^2 \int [\rho(g^{abc})_{;cd} - \rho(g^{aec})_{;ce}\delta^{b}{}_{d}]\, \delta\Gamma^{d}{}_{ab}\, d^9x$$

$$+ \int \rho g^{abc}[R_{db}\, \delta\Gamma_{dac} + R_{ad}\, \delta\Gamma^{d}{}_{bc}]\, d^9x = 0$$

or rearranging the dummy indices,

$$\int [\rho(g^{abc})_{;cd} - \rho(g^{aec})_{;ce}\delta^{b}{}_{d}$$

$$+ \rho g^{abc}(R_{dc} + R_{cd}]\, \delta\Gamma^{d}{}_{ab}\, d^9x = 0$$

Thus, set

$$(g^{abc})_{;cd} - (g^{aec})_{;ce}\delta^{b}{}_{d} + g^{abc}(R_{dc} + R_{cd}) = 0$$

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REFERENCES

Barnabei, M., Brini, A., and Rota, G. (1985). *Journal of Algebra*, **96**, 120–160.

Bergmann, P. G. (1957). *Physical Review*, **107**, 624.

Borowiec, A. (1988). *International Journal of Theoretical Physics*, **28**, 1229–1232.

Budinich, P., and Trautman, A. (1988). *The Spinorial Chessboard*, Springer-Verlag, Berlin.

Busemann, H. (1942). *Metric Methods in Finsler Spaces and in the Foundations of Geometry*, Princeton University Press, Princeton, New Jersey.

Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford.

Collins, P. D. B., Martin, A. D., and Squires, E. J. (1989). *Particle Physics and Cosmology*, Wiley, New York.

Cvitanovic, P., and Kennedy, A. D. (1982). *Physica Scripta*, **26**, 5–14.

Finkelstein, S. R. (1987). *International Journal of Theoretical Physics*, **27**, 251–272.

Finkelstein, D., Finkelstein, S. R., and Holm, C. (1985). *International Journal of Theoretical Physics*, **25**, 441–463.

Finkelstein, D., Finkelstein, S. R., and Holm, C. (1987). *Physics Review Letters*, **59**, 1265–1266.

Friedman, M. (1983). *Foundations of Space-Time Theories*, Princeton University Press, Princeton, New Jersey.

Hawking, S. W., and Ellis, G. F. R. (1989). *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge.

Holm, C. (1986). *International Journal of Theoretical Physics*, **25**, 1209–1213.
Holm, C. (1987). *The hyperspin structure of Einstein universes and their neutrino spectrum*, Ph.D. Thesis, School of Physics, Georgia Institute of Technology, Atlanta, Georgia.
Holm, C. (1989). *International Journal of Theoretical Physics*, **29**, 23–36.
Kilmister, C. W. (1973). *General Theory of Relativity*, Pergamon Press, Oxford.
Mantke, W. (1989). *Spin and gravity*, M.S. Thesis, School of Physics, Georgia Institute of Technology, Atlanta, Georgia.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco.
Penrose, R., and Rindler, W. (1986). *Spinors and Space-Time*, Cambridge University Press, Cambridge.
Rindler, W. (1979). *Essential Relativity: Special, General, and Cosmological*, Springer-Verlag, Berlin.
Schutz, B. F. (1986). *A First Course in General Relativity*, Cambridge University Press, Cambridge.
Stephani, H. (1982). *General Relativity: An Introduction to the Theory of the Gravitational Field*, Cambridge University Press, Cambridge.
Straumann, N. (1984). *General Relativity and Relativistic Astrophysics*, Springer-Verlag, Berlin.
Wald, R. M. (1984). *General Relativity*, University of Chicago Press, Chicago.
Weinberg, S. (1984). *Physics Letters*, **138B**, 47–51.
Weyl, H. (1946). *The Classical Groups*, Princeton University Press, Princeton, New Jersey.